Conditional statistics in scalar turbulence: Theory versus experiment

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We consider turbulent advection of a scalar field $T(\mathbf{r})$, passive or active, and focus on the statistics of gradient fields conditioned on scalar differences $\Delta T(R)$ across a scale R. In particular we focus on two conditional averages $\langle\nabla^2 T|\Delta T(R)\rangle$ and $\langle|\nabla T|^2|\Delta T(R)\rangle$. We find exact relations between these averages, and with the help of the fusion rules we propose a general representation for these objects in terms of the probability density function $P(\Delta T,R)$ of $\Delta T(R)$. These results offer a way to analyze experimental data that is presented in this paper. The main question that we ask is whether the conditional average $\langle\nabla^2 T|\Delta T(R)\rangle$ is linear in ΔT. We show that there exists a dimensionless parameter which governs the deviation from linearity. The data analysis indicates that this parameter is very small for passive scalar advection, and is generally a decreasing function of the Rayleigh number for the convection data. [S1063 -651X(96)11312-X]

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I. INTRODUCTION

The equations of motion in fluid mechanics, be they for the velocity field $\mathbf{u}(\mathbf{r},t)$ or for a scalar field like the temperature $T(\mathbf{r},t)$, contain interaction terms like $\mathbf{u}\cdot\nabla\mathbf{u}$ or $\mathbf{u}\cdot\nabla T$, and dissipative terms like $\nu\nabla^2\mathbf{u}$ or $\kappa\nabla^2 T$, with ν and κ being the kinematic viscosity and the scalar diffusivity, respectively. Accordingly, when one attempts to derive a theory of correlation functions of the field or of field differences across a length scale R, one runs into mixed correlation functions of the Laplacian of the field with the field itself. The Laplacian introduces a dependence on dissipative scales, and it is not obvious how to construct a closed theory in terms of correlation functions that depend on ''inertial'' distances only. At this point there are two fundamentally different approaches that we want to expose first. We exemplify these approaches in the context of turbulent advection, but similar considerations also apply to Navier-Stokes turbulence.

One fundamental strategy, which is the more usual one, is to consider the structure functions of the field differences. Denoting $\Delta T(\mathbf{r},\mathbf{r}',t)\equiv T(\mathbf{r}',t)-T(\mathbf{r},t)$, the structure functions $S_n(R,t)$ are defined by

$$S_n(R,t)=\langle[\Delta T(\mathbf{r},\mathbf{r}',t)]^n\rangle,\quad R=|\mathbf{R}|\equiv|\mathbf{r}'-\mathbf{r}|,\tag{1}$$

where the homogeneity and isotropy of the ensemble were assumed. Using the advection equation

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T=\kappa\nabla^2 T,\tag{2}$$

one can derive the equation of motion for $S_n(R,t)$:

$$\frac{\partial S_n(R,t)}{\partial t}+D_n(R,t)=J_n(R,t),\tag{3}$$

where

$$D_n(R,t)=2n\langle[\mathbf{u}(\mathbf{r},t)\cdot\nabla\Delta T(\mathbf{r},t)][\Delta T(\mathbf{r},\mathbf{r}',t)]^{n-1}\rangle,\tag{4}$$

$$J_n(R,t)=-2n\kappa\langle\nabla^2 T(\mathbf{r},t)[\Delta T(\mathbf{r},\mathbf{r}',t)]^{n-1}\rangle.\tag{5}$$

In the stationary state $S_n(R,t)\to S_n(R)$, $D_n(R,t)\to D_n(R)$, and $J_n(R,t)\to J_n(R)$, and we have a ''balance equation'' $D_n(R)=J_n(R)$.

The obvious advantage of this approach is that it involves objects that depend on one coordinate R only. On the other hand the analysis of the balance equation requires a theory of the *viscous* term J_n which involves a correlation between the Laplacian of the field and field differences. Even when R is within the inertial range, one cannot get rid of $J_n(R)$. The limit $\kappa\to 0$ does not help; J_n has a finite value in this limit.

A second fundamental approach which avoids this difficulty [1] employs multipoint correlation function of field differences. Starting from the same field differences $\Delta T(\mathbf{r}_1,\mathbf{r}_1',t)$, one defines the correlation function

$$\mathcal{F}_n=\mathcal{F}_n(\mathbf{r}_1,\mathbf{r}_1';\ldots\mathbf{r}_n,\mathbf{r}_n';t)\equiv\langle\Delta T_1,\ldots\Delta T_n\rangle,\tag{6}$$

which depends on n fields $\Delta T_j\equiv\Delta T(\mathbf{r}_j,\mathbf{r}_j',t)$, $2n$ coordinates $\mathbf{r}_j$, $\mathbf{r}_j'$ and time t. The equation of motion for $\mathcal{F}_n$ looks superficially similar to Eq. (3):

$$\frac{\partial\mathcal{F}_n}{\partial t}+\mathcal{D}_n=\mathcal{J}_n,\tag{7}$$

where

$$\begin{aligned}\mathcal{D}_n&=\mathcal{D}_n(\mathbf{r}_1,\mathbf{r}_1';\ldots\mathbf{r}_n,\mathbf{r}_n';t)\\&=\sum_{j=1}^{n}\langle\Delta T_1\ldots[\mathbf{u}(\mathbf{r}_j')\cdot\nabla_{j'}T(\mathbf{r}_j')\\&\quad-\mathbf{u}(\mathbf{r}_j)\cdot\nabla_j T(\mathbf{r}_j)]\ldots\Delta T_n\rangle,\end{aligned}\tag{8}$$

$$\begin{aligned}\mathcal{J}_n&=\mathcal{J}_n(\mathbf{r}_1,\mathbf{r}_1';\ldots\mathbf{r}_n,\mathbf{r}_n';t)\\&=\kappa\sum_{j=1}^{n}(\nabla_j^2+\nabla_{j'}^2)\langle\Delta T_1\ldots\Delta T_j\ldots\Delta T_n\rangle.\end{aligned}\tag{9}$$

In the stationary state we again face a ''generalized balance equation'' $\mathcal{D}_n = \mathcal{J}_n$. In fact, there is a fundamental difference between the two approaches. In the present case one can analyze the generalized balance equation, keeping all the separations within the inertial interval of scales. Then we can take the limit $\kappa \to 0$ and show [1] that the dissipative term $\mathcal{J}_n$ vanishes in the limit. We remain in this limit with a homogeneous equation $\mathcal{D}_n = 0$, without any Laplacian terms. The advantage is that in principle we have a complete theory without the need of additional input. The obvious disadvantage of this approach is that we have functions of many variables. Nevertheless, this approach turned out to be very useful in the context of Kraichnan's model for passive scalar advection [2], where the homogeneous equation can be turned into a linear partial differential equation for the correlation functions $\mathcal{F}_n$. However even in this simplest possible case the difficulty incurred by having functions of many variables led to contradicting arguments about the relevant physical solution. A number of groups attempted a perturbative solution around tractable limits [3,4], and found results that were in contradiction with numerical simulations and other theoretical arguments [5,6], including nonperturbative ones [7]. The final answer is not yet at hand.

It is thus obviously useful to find ways to analyze the simpler version in which we have one variable only, but a mixture of inertial and dissipative scales contributing to the correlation function J_n. To proceed we need additional information. One possible way of inputting this information is in the language of conditional averages. To see this, consider the mean value of the $\kappa \nabla^2 T$ condition on ΔT:

$$H(\Delta T, R) \equiv \kappa \langle \nabla^2 T(\mathbf{r}) | \Delta T(\mathbf{r}, \mathbf{r}') \rangle. \tag{10}$$

Using this definition, we rewrite $J_n(R)$ as

$$J_n(R) = -2n \int d\Delta T P(\Delta T, R) [\Delta T]^{n-1} H(\Delta T, R), \tag{11}$$

where $P(\Delta T, R)$ is the probability density function (pdf) to find a temperature difference ΔT across a separation R. It was proposed by Kraichnan [8] that the conditional average $H(\Delta T, R)$ exhibits in his model a very simple functional dependence on ΔT and on R, i.e.,

$$H(\Delta T, R) = \frac{-J_2 \Delta T(R)}{4 S_2(R)}. \tag{12}$$

If this were true, $J_n(R)$ could immediately be written in terms of structure functions,

$$J_n(R) = \frac{n J_2 S_n(R)}{2 S_2(R)}. \tag{13}$$

This again closes the theory upon itself, allowing one to proceed in terms of objects that depend on inertial scales only. In the Kraichnan model, D_n can also be written in terms of $S_n(R)$ [see Eq. (25) below] and one can therefore find the scaling exponents that characterize the structure functions. Unfortunately there is still no derivation of the ansatz (12). There are indications that it is obeyed; numerical simulations of the Kraichnan model support it rather convincingly [5]. In addition, it was shown in [9], on the basis of experimental data analysis, that this form of the conditional average is obeyed in a context that is much wider than the Kraichnan model. Experiments on turbulent advection were analyzed, and good agreement with Eq. (12) was demonstrated. The aim of this paper is to present further theoretical and data analysis in this direction. We want to understand what can be said on conditional averages in terms of fundamental theory, and how to analyze experimental data intelligently to probe these important quantities.

It should be pointed out that although we focus in this paper on turbulent advection, similar considerations are also important in the context of Navier-Stokes turbulence. Also in that case, two fundamental strategies to develop a statistical theory are open to us. The second strategy is even more tempting in that context. In the first approach one obtains objects depending on one coordinate, but a hierarchy of equations relating different order (in powers of the velocity field) correlation functions. The second strategy gives a theory involving many coordinates, but in which we can also neglect the viscous term, obtaining homogeneous equations $\mathcal{D}_n = 0$ that involve only one order of correlation functions. We are not going to explore this issue further, and refer the reader to [1] for more details.

The paper is organized as follows: In Sec. II we present theoretical considerations that relate conditional averages with the probability density function. The fusion rules are employed to develop a general representation of the conditional average (10) in terms of the pdf of $\Delta T(R)$. It is shown that in general $H(\Delta T, R)$ can be written as an expansion in noninteger powers of ΔT, with the first term being linear, and with dimensionless coefficients that are denoted $a_0, a_1 \ldots$, see Eqs. (22) and (23). In Sec. III we analyze experimental data of passive and active scalar advection, with the aim of understanding whether the linear term in our expansion is leading, and whether the rest of the series is unnecessary. We offer conclusions in Sec. IV: it turns out that for passive scalar advection the linear fits are excellent, whereas in the case of active convection the linear form appears to fit the data extremely well for high values of the Rayleigh numbers (Ra). For lower values of Ra there are significant nonlinear contributions, and we show that the proposed method of data analysis offers excellent fits to the data.

II. CONDITIONAL AVERAGES AND THE RELATIONS BETWEEN THEM

A. Conditional average of the dissipation field

In addition to the conditional average (10), we will consider the average of the scalar dissipation field $\kappa |\boldsymbol{\nabla} T|^2$ conditioned on ΔT:

$$G(\Delta T, R) = \kappa \langle |\boldsymbol{\nabla} T|^2 | \Delta T(\mathbf{r}, \mathbf{r}') \rangle. \tag{14}$$

This conditional average appears naturally in the analysis of $J_n(R)$. For space homogeneous statistics one can move one gradient around in the definition (5) and, for R in the inertial range, obtain

$$J_n(R) \sim -2n(n-1) \kappa \langle |\boldsymbol{\nabla} T(\mathbf{r})|^2 [\Delta T(\mathbf{r}, \mathbf{r}')]^{n-2} \rangle. \tag{15}$$

Accordingly, we can write a second equation in terms of the probability density function

$$J_n(R) = -2n(n-1)\int d\Delta T P(\Delta T,R)[\Delta T]^{n-2}G(\Delta T,R). \tag{16}$$

By equating Eqs. (16) and (11), we find an infinite set of integral constraints on the conditional averages. This implies that the two conditional averages H and G must be universally related in order to satisfy these constraints for any value of n. The required relationship involves the pdf $P(\Delta T,R)$ and has the following form:

$$H(\Delta T,R)P(\Delta T,R) = -\frac{\partial}{\partial \Delta T}[G(\Delta T,R)P(\Delta T,R)], \tag{17}$$

as can be checked by direct substitution. A formula of this form has been discussed before in [10].

B. Fusion rules and their consequences

An additional constraint on the conditional averages can be obtained using the "fusion rules" that were derived recently. These rules serve to find relationships between the two fundamental approaches described in Sec. I. Specifically, the fusion rules address the asymptotic properties of $\mathcal{F}_n$ when a group of p points, $p<n-1$ tend toward a common point $\mathbf{r}_0$ ($|\mathbf{r}_i-\mathbf{r}_0|\sim\rho$ for all $i\leqslant p$), while all the other coordinates remain at a larger distance R from $\mathbf{r}_0$ ($|\mathbf{r}_i-\mathbf{r}_0|\sim R$ for $i>p$, and $R\gg\rho$). For our particular purposes we need to write $J_n(R)$ as the result of the following fusion process:

$$J_n(R) = -2n\kappa \lim_{\mathbf{r}_i\to\mathbf{r}_0} \lim_{\mathbf{r}'_i\to\mathbf{r}_0+\mathbf{R}} \boldsymbol{\nabla}^2_{r_1}\mathcal{F}_n(\mathbf{r}_1,\mathbf{r}'_1;\ldots\mathbf{r}_n,\mathbf{r}'_n). \tag{18}$$

The fusion rules that should be used in such cases were displayed in great detail in [1] in the context of Navier-Stokes turbulence. They also apply identically to this case. Basically it was shown that all the fusions without gradients in this case have regular limits, relating $\mathcal{F}_n$ with S_n. The fusions with gradients require special care of the limit $\mathbf{r}_{12}\equiv\mathbf{r}_1-\mathbf{r}_2\to 0$. The intermediate result, for R in the inertial range, is

$$J_n(R) \sim -2n\kappa \lim_{\mathbf{r}_{12}\to 0} \boldsymbol{\nabla}^2_{r_1} S_2(r_{12})S_n(R)/S_2(R). \tag{19}$$

In evaluating this quantity we interpret the limit $\mathbf{r}_{12}\to 0$ as a limit $r_{12}\to\eta$. This seems natural for large Peclet numbers when $\eta\to 0$. It is important however to stress that there is a hidden assumption here. We expect the function $\mathcal{F}_{2n}$ to change its analytic behavior as a function of r_{12}. This change occurs at the viscous crossover scale η. The issue is whether this crossover scale is n and R independent. That this is so has been *proven* for Kraichnan's model of turbulent advection [11], and that this is *not* so has been proven for Navier-Stokes turbulence [1]. We believe that this is more generally true in scalar advection due to the linearity of the equation of motion (2), independently of the statistical properties of the driving velocity field. The experimental results analyzed in [9] strongly indicate that this is the case in a wide context of turbulent scalar fields. With this in mind we write

$$J_n(R) \sim -2\kappa n[\nabla^2_{r_{12}} S_2(r_{12})|_{r_{12}=\eta}]S_n(R)/S_2(R). \tag{20}$$

Using the fact that the mean of the scalar dissipation field, denoted $\bar{\epsilon}$, is evaluated as $2\bar{\epsilon}\sim\kappa[\nabla^2_\rho S_2(\rho)|_{\rho=\eta}]$, and also the fact that in the inertial range $J_2(R)=-4\bar{\epsilon}$, we write

$$J_n(R) = \frac{nC_nJ_2S_n(R)}{2S_2(R)}, \tag{21}$$

where C_n is an as yet unknown dimensionless coefficient, but $C_2=1$. Equation (21) was suggested for Kraichnan's model in [8] and derived in [6]. We proposed in [9] that it holds in a much wider context, and showed experimental data in support.

Having result (21), we see that the scaling exponent of $J_n(R)$ is fixed as $\zeta_n-\zeta_2$. One way to understand this is to assume that indeed Eq. (12) is always valid, and to use it in Eq. (11) to derive this scaling law. In this case Eq. (21) is recovered with the constraint that the coefficients C_n are all unity, and in particular n-independent. However, we need to allow for the possibility that Eq. (12) is incorrect, and find alternative representations of the conditional average that agree with the scaling law (21). This is the subject of the next subsection.

C. Series expansion of the conditional average

Let us reject for the time being the possibility that $H(\Delta T,R)$ is proportional to $\Delta T(R)$ with a coefficient depending only on R. Alternatively, let us consider the following model expression for $H(\Delta T,R)$:

$$H(\Delta T,R) = -\frac{J_2}{4S_2(R)}\frac{\hat{\mathcal{L}}\{\Delta T P(\Delta T,R)\}}{P(\Delta T,R)}. \tag{22}$$

Here we introduce the dimensionless operator $\hat{\mathcal{L}}$ acting on the variable $\Delta T(R)$ as a sum of differential operators:

$$\hat{\mathcal{L}} = \sum_{p=0}^{\infty}\frac{a_p}{p!}\left[\frac{\partial}{\partial\Delta T}\right]^p(\Delta T)^p. \tag{23}$$

In this representation there is the freedom of a countable set of dimensionless coefficients a_p.

From the dimensional point of view, $H(\Delta T,R)$ in Eq. (22) is of the order of ΔT, but it has a more complicated functional dependence on ΔT and R, expressed in terms of the pdf $P(\Delta T,R)$ with the help of operator $\hat{\mathcal{L}}$. Computing $J_n(r)$ with H given by Eq. (22), one obtains Eq. (21), as one should, but with coefficients C_n given by

$$C_n = \sum_{p=0}^{n-1}\binom{n-1}{p}(-1)^p a_p. \tag{24}$$

Here

$$\binom{n-1}{p} = \frac{(n-1)!}{p!(n-p-1)!}$$

are binomial coefficients. We have one obvious constraint, i.e., $C_2 = a_0 - a_1 = 1$. One sees that by an appropriate choice of a_p, an arbitrary dependence of C_n on n is possible.

To exemplify the consequences of this extra freedom, we will next analyze the implications it has on the scaling exponents of the Kraichnan model of passive scalar convected by an infinitely fast Gaussian velocity field. In this case the term $D_n(R)$ is known exactly,

$$D_n(R) = \frac{B}{R^{d-1}} \frac{d}{dR} R^{d-1+\zeta_h} \frac{d}{dR} S_n(R), \tag{25}$$

where ζ_h is the parameter of the model, $0 < \zeta_h < 2$, and B is a dimensional constant. Using the balance equation and writing $S_n(R) \propto R^{\zeta_n}$, one computes

$$\zeta_n = \tfrac{1}{2}\sqrt{(d-\zeta_2)^2 + 2nd\zeta_2 C_n} - \tfrac{1}{2}(d-\zeta_2). \tag{26}$$

If all coefficients $a_{(m\geqslant 1)}$ are zero, then $C_n = 1$ and we have Kraichnan's conjecture for ζ_n:

$$\zeta_n = \tfrac{1}{2}\sqrt{(d-\zeta_2)^2 + 2nd\zeta_2} - \tfrac{1}{2}(d-\zeta_2), \tag{27}$$

which dictates the ''square-root'' asymptotic behavior $\zeta_n \to \sqrt{nd\zeta_2/2}$ in the limit $n \to \infty$. Assume now that only a_0 and a_1 are nonzero. From Eq. (24), $C_n = a_0 - (n-1)a_1$, or

$$C_n = 1 - (n-2)a_1, \tag{28}$$

and if we use this result in the Kraichnan model we obtain

$$\zeta_n = \tfrac{1}{2}\sqrt{(d-\zeta_2)^2 + 2nd\zeta_2[1-(n-2)a_1]} - \tfrac{1}{2}(d-\zeta_2). \tag{29}$$

Now the asymptotics of ζ_n in n are linear: $\zeta_n \propto n\sqrt{-a_1}$ for $n \to \infty$. Notice that in this case, a_1 has to be negative. It is interesting to note that the assumption that only the first three coefficients a_0, a_1, and a_2 are nonzero would lead to the conclusion that $\zeta_n \propto n^{3/2}$, which is not allowed in view of the Hoelder inequalities for the scaling exponents. Similarly, one cannot truncate series (23) at any higher term. Hence only three possibilities are allowed for this representation of the conditional average:

(i) Only a_0 is nonzero, and we have Kraichnan's exponents (27). In this case we expect to find a *linear* law

$$H(\Delta T, R) = -\frac{J_2}{4S_2(R)} \Delta T(R). \tag{30}$$

(ii) Only a_0 and a_1 are nonzero, and we have the exponent (29). Note that Eq. (28) determines the coefficients C_n in this case, and the magnitude of the dimensionless parameter a_1 measures the deviation of C_n from unity. We will see below that in all high Re data a_1 seems to be smaller than 10^{-2}.

(iii) There is an infinite set of nonzero coefficients a_p.

It is interesting to ask whether one can come up with an example of infinitely many coefficients a_p without violating any general requirement about the scaling exponents. In fact, this can be easily done. For example, choose a_p of the following form:

$$a_p = \sum_s \beta_s[1-\exp(-\alpha_s)]^p + \delta_{p0}\mu - \delta_{p1}\nu, \tag{31}$$

and substitute in Eq. (24). The result is

$$C_n = \mu + (n-1)\nu + \sum_{p=0}^{n-1} \binom{n-1}{p} \sum_s \beta_s (e^{-\alpha_s} - 1)^p. \tag{32}$$

Finally it gives

$$C_n = \mu + (n-1)\nu + \sum_s \beta_s \exp[-(n-1)\alpha_s], \tag{33}$$

wich satisfies the constraint

$$C_2 = \mu + \nu + \sum_s \beta_s \exp(-\alpha_s) = 1. \tag{34}$$

In the limit $n \gg \max[1/\alpha_s]$, we find

$$\zeta_n = \tfrac{1}{2}\sqrt{(d-\zeta_2)^2 + 2nd\zeta_2[\mu + (n-1)\nu]} - \tfrac{1}{2}(d-\zeta_2). \tag{35}$$

This form again has an asymptotic linear dependence of ζ_n on n, but for intermediate values of n these exponents differ significantly from Eq. (29). We do not ascribe particular importance to this result, and exhibit it only to show that, to satisfy the consequences of the fusion rules, in general we have considerable freedom in the functional dependence of ζ_n on n.

It is important to understand now that the series (22) is actually an expansion in *noninteger powers* of ΔT. As such, it is fundamentally different from the series proposed in Ref. [5], which is in *integer* powers. The noninteger powers are dictated by the functional form of the pdf $P(\Delta T, R)$, which in general is nonanalytic. In order to see this clearly, we consider, for example, a form $P(\Delta T, R)$ that has been found [12] to fit very well the experimental data for turbulent temperature fluctuations. For different separations R, the pdf is described by the following stretched-exponential form:

$$P(\Delta T, R) = C(R)\exp[-\alpha(R)|\Delta T|^{\beta(R)}]. \tag{36}$$

Substitution of this into Eq. (22) gives a series in *noninteger* powers of ΔT which originate from the differentiation of $(\Delta T)^\beta$. Any attempt to reexpand the series in $(\Delta T)^{m\beta}$ for *noninteger* β in *integer* powers of ΔT leads unavoidably to a series with zero radius of convergence.

III. IS THE CONDITIONAL AVERAGE $H(\Delta T, R)$ LINEAR OR NONLINEAR IN ΔT?

As we already saw, the present state of the theory does not allow an *ab initio* determination of the functional dependence of the conditional average H on ΔT. Accordingly, we now turn to analyzing experimental data to shed light on this issue. As explained, the conditional average $H(\Delta T, R)$ is linear in ΔT if and only if all the coefficients except a_0 are zero [i.e. possibility (i) discussed in Sec. II]. For the other two cases, $H(\Delta T,R)$ is a nonlinear function of ΔT. In earlier work [9], we found that $H(\Delta T,R)$ is close to a linear function of ΔT, which implies that $a_p, p\neq 0$ are small compared to a_0. To make more quantitative statements, we will perform a further analysis of the experimental data under the assumption that a_0 and a_1 are nonzero. We will see that this form fits the data extremely well, and that the coefficient a_1 is always small, and it appears to become smaller when the Reynolds number is increasing and when the chosen separation goes into the bulk of the inertial interval. Taking a_0 and a_1 as the only nonzero coefficients, we find

$$H(\Delta T,R)=\frac{-J_2\Delta T}{4S_2(R)}\left\{(a_0+2a_1)+a_1\Delta T\frac{\partial[\ln P(\Delta T,R)]}{\partial \Delta T}\right\}. \tag{37}$$

Thus the coefficient a_1 indeed measures how nonlinear H is. Using form (36) for the pdf $P(\Delta T,R)$, H can be rewritten as

$$H(\Delta T,R)=\frac{-J_2\Delta T}{4S_2(R)}[(1+3a_1)-a_1\alpha(R)\beta(R)|\Delta T|^{\beta(R)}], \tag{38}$$

in which $a_0=1+a_1$ is used. When $P(\Delta T,R)$ is asymmetric, a more general form with different α and β for $\Delta T>0$ and $\Delta T<0$ has to be used. Note that if we measure ΔT in units of its standard deviation (as we do below in the data analysis), then the combination of parameters $a_1\alpha(R)\beta(R)$ gives a direct measure of the importance of the nonlinear correction in this equation for $\Delta T=1$.

A. Linear fits

We begin the discussion of experimental data by demonstrating that in the case of passive scalar advection the linear form of the conditional average $H(\Delta T,R)$ is observed to high precision. First we examine the theoretical prediction (21). The results show that to a good accuracy $C_n\approx 1$ for all n and R [9].

We use temperature data measured in the wake of a heated cylinder [13]. Air of speed 5 m/s flowed past a heated cylinder of diameter 19 mm (Reynolds number 9.5×10^4). The temperature was measured at a fixed point downstream of the cylinder on the wake centerline. The cylinder was heated so slightly that the buoyancy term was unimportant and the temperature acted as a passive scalar. Temperature was measured as a function of time, and here we use the standard Taylor hypothesis that surrogates time derivatives for space derivatives. In doing so we made sure that the viscous scales are properly resolved in this data set. In Fig. 1 we display $J_{2n}(R)/(2n\kappa)$ as a function of $(2\kappa)^{-1}J_2S_{2n}(R)/S_2(R)$ for n varying from 2 to 6, and for various R values in the inertial range. We see that all the points fall on a line whose slope is unity to high accuracy, and whose intercept (in log-log plot) is very closely zero. As was pointed out in [9], this good agreement is a confirmation of the validity of the fusion rules. It should be stressed that individual tests at various values of n as a function of R corroborate the same conclusion, i.e., Eq. (21) is supported by the experimental data, with C_{2n} being near unity. The most sensitive test of the alleged constancy of the coefficients C_{2n} is obtained by dividing $J_{2n}(R)$ by $nJ_2S_{2n}(R)/S_2(R)$ for all the available values n and R. The result of such a test is shown in Fig. 2.

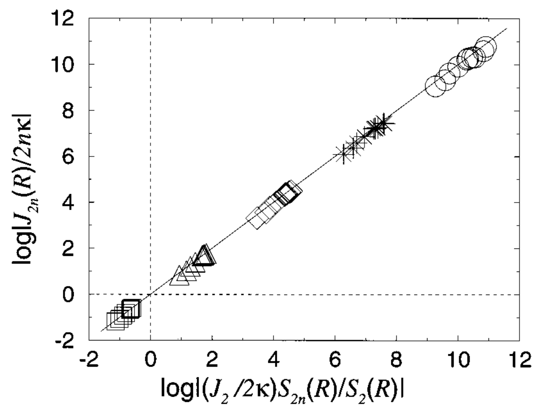

FIG. 1. A plot of $\ln|J_{2n}(R)/(2n\kappa)|$ vs $\ln|(2\kappa)^{-1}J_2S_{2n}(R)/S_2(R)|$ for $n=2$ (squares), 3 (triangles), 4 (diamonds), 5 (stars), and 6 (circles) and R in the inertial range. The data are taken from Yale [13]. The line (slope 1 and intercept 0) is not a fit but is the theoretical expectation (21) with $C_n=1$. The logarithms are to base e.

We see that all the measured values of C_{2n} are concentrated within the interval (0.75,1) for separations within the inertial interval. Considering the fact that the quantities themselves vary in this region over five orders of magnitude, we interpret this as a good indication for the independence of C_{2n} of R and n. The R independence is very clear, and is a direct test of the fusion rules. The weak n dependence seems to indicate that C_{2n} decreases slightly with n; this may arise from the limited accuracy of the data. We are reluctant to make a strong claim about the accuracy of tenth- or twelfth-order structure functions. If we accept for now the evidence that the coefficients C_{2n} in Eq. (21) are n independent, it must also imply that the conditional average $H(\Delta T,R)$ is linear in ΔT.

In Fig. 3 we present results from the same data set that was used above. We show the conditional average $H(\Delta T,R)$ as a function of $\Delta T(R)$ for various values of R.

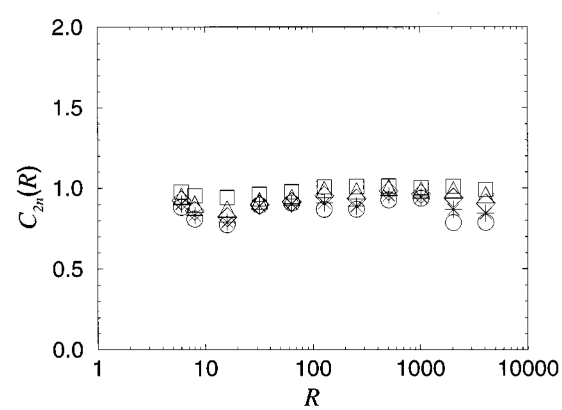

FIG. 2. A detailed test of the coefficient C_{2n}; see text for details. The symbols are the same as in Fig. 1. The small systematic decrease of C_{2n} with n may be due to insufficient accuracy at the tails of the probability density function which becomes more important at large values of n.

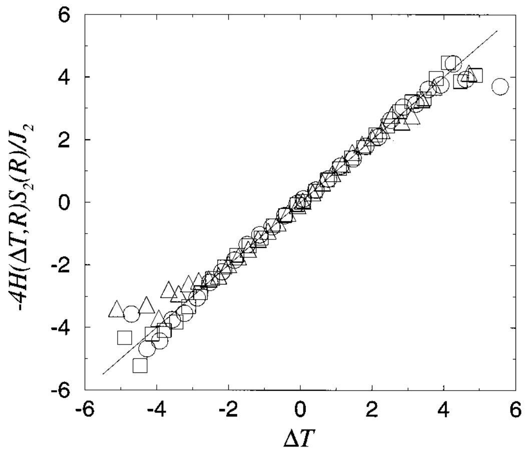

FIG. 3. The conditional average $H(\Delta T,R)$ measured from the Yale data [10] normalized by the measured value of $-J_2/4S_2(R)$ as a function of $\Delta T(R)$ for three different values of R measured in units of the sampling time. The different R values are designated by triangles ($R=16$), squares ($R=128$), and circles ($R=1024$), respectively.

The line passing through the data points is not a fit, but rather the line required by Eq. (30). We note that points belonging to different values of R fall on the same line, indicating that indeed the conditional average H is a function of $\Delta T(R)$ times a function of R, and that we identified correctly the function of R as $-J_2/4S_2(R)$.

B. Nonlinear fits

As explained, the linearity of the conditional average of H in ΔT [Eq. (30)] was not derived from first principles. We therefore proceed now to see whether the more general form (38) is supported by the data, and whether we can bound from above the values of the parameter a_1. To estimate a_1 from experimental data, we first estimate $\alpha(R)$ and $\beta(R)$ from the pdf's evaluated from data using Eq. (36) then plot $-4H(\Delta T,R)S_2(R)/[J_2\Delta T]$E versus $|\Delta T|^{\beta(R)}$. The intercept is given by $1+3a_1$. To study how well Eq. (38) can represent the data, we subsitute the estimated value of a_1 into Eq. (38) and compare it with the experimental data. Since the passive scalar data shown in Fig. 3 agree so well with the linear ansatz, we first discuss a case that offers a more stringent test of the form (38). To this aim we consider data taken from convective turbulence. In this case the temperature is an active rather than a passive scalar. The data are taken from the well documented Chicago experiment [14,15]. The experiment was performed in a cylindrical box of helium gas heated from below, and Ra can be as high as 10^{15}. The box has a diameter of 20 cm and a height of 40 cm. The temperature at the center of the box was measured as a function of time, and we use the same Taylor hypothesis to surrogate time for the spatial coordinate. Figure 4 display the conditional average $H(\Delta T,R)$ computed from these data for three different values of Ra, with R measured in units of the sampling time. We see that Eq. (38) is always a good form for describing the experimental data. It is interesting to examine how the nonlinearity in the conditional average depends on Ra and on the value of R. In Table I we present a compila-

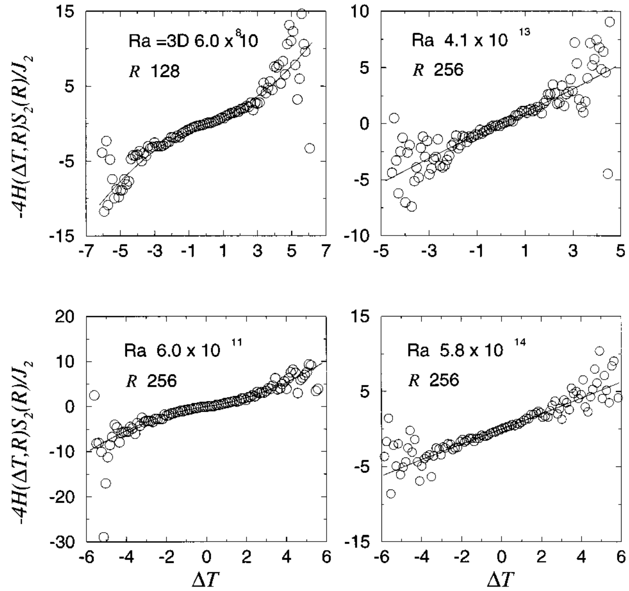

FIG. 4. The conditional average $H(\Delta T,R)$ as a function of ΔT for the turbulent convection data. Shown are representative fits of formula (38) at four values of Ra with the separation R measured in units of sampling time using Taylor hypothesis. One sees that the linear fit becomes better as Ra increases; see Table I for a quantitative confirmation

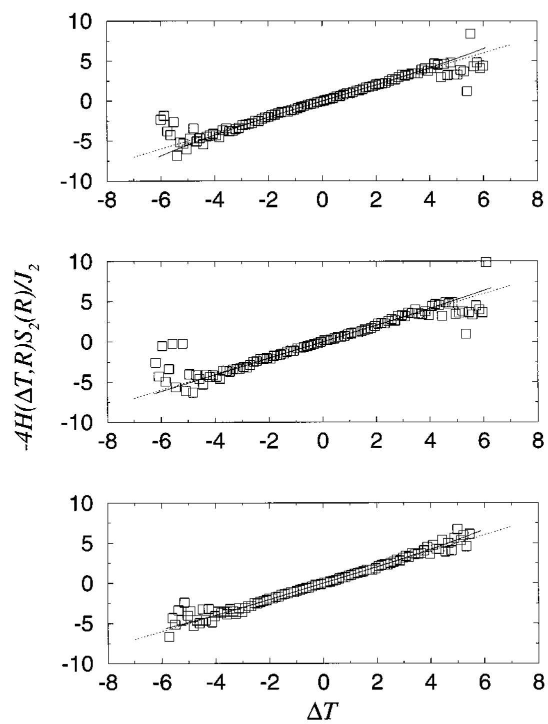

FIG. 5. The conditional average $H(\Delta T,R)$ as a function of ΔT for the passive scalar data at three values of the separation R (from top to bottom $R=128$, 256, and 512), measured in units of sampling time. The nonlinear fits (solid lines) are indistinguishable from the linear ones (dotted lines) in the bulk of the inertial range, and see Table II for a quantitative confirmation

TABLE I. Fitted parameters for the turbulent convection data, with the Rayleigh number spanning the range 6×10^8 to 5.8×10^{14}. The parameters α and β characterize the probability density function $P(\Delta T,R)$, and a_1 is the coefficient of the first nonlinear contribution to the expansion of the conditional average $H(\Delta T,R)$ in ΔT. The separation R is measured in units of the sampling time. The value of the parameter a_1 measures the deviation of the coefficients C_n from unity, cf. Eq. (28). The combination $a_1\alpha\beta$ measures the deviation of the conditional average from the linear fit at $\Delta T=1$, cf. Eq. (38).

Ra	R	8	16	32	64	128	256	512	1024	2048
6×10^8	β	0.54	0.64	0.71	0.95	1.06	1.31	1.74	1.84	—
	$100a_1$	−40.3	−22.8	−16.5	−12.9	−8	−17.6	−7.3	−16.7	—
	$100a_1\alpha\beta$	−116	−57	−37.2	−24.9	−14.3	−14.3	−4.4	−10.0	—
4×10^9	β	0.52	0.67	0.85	1.05	1.19	1.41	1.81	2.00	
	$100a_1$	−24.8	−14.6	−12.8	−5.4	−7.2	−6.3	−3.1	−16.4	
	$100a_1\alpha\beta$	−68	−33.4	−22.4	−8.7	−9.0	−8.1	−2.6	−10.0 a	
7.3×10^{10}	β		0.61	0.68	0.80	0.91	1.15	1.38	1.55	1.68
	$100a_1$	—	−9.1	−7.3	−7.1	−1.1	−16.3	−12.1	−14.8	−5.7
	$100a_1\alpha\beta$	—	−24.5	−17.6	−12.1	−3.3	−15.5	−9.5	−11.1	−3.0
6×10^{11}	β		0.60	0.65	0.72	0.84	0.96	1.19	1.50	1.43
	$100a_1$		−6.9	−2.8	−14.0	−12.9	−10.9	−13.7	−4.4	−14.2
	$100a_1\alpha\beta$		−18.8	−9.1	−27.1	−20.6	−71.0	−15.7	−4.7	−11.6
6.7×10^{12}	β	—	0.58	0.64	0.72	0.83	0.94	1.00	1.25	1.45
	$100a_1$		−7.1	−20.4	−17.5	−8.2	−6.7	−3.9	−7.1	−8.1
	$100a_1\alpha\beta$		−22.9	−48.1	−34.9	−14.4	−12.4	−9.6	−9.5	−7.6
4.1×10^{13}	β		0.61	0.68	0.77	0.86	0.88	1.15	1.26	1.42
	$100a_1$		−2.9	−1.5	−0.5	−2.3	−1.2	−0.7	−30.7	−10.5
	$100a_1\alpha\beta$		6.4	−4.7	−1.9	−6.0	−2.4	−2.4	−22.5	−9.4
5.8×10^{14}	β		0.59	0.63	0.69	0.85	0.93	1.05	1.31	1.56
	$100a_1$		−4.7	−2.6	−4.5	−5.3	−0.4	−2.5	−8.7	−3.1
	$100a_1\alpha\beta$		−10.9	−4.7	−6.0	−4.6	−1.0	−4.3	−8.9	−2.8

tion of the best fits for the parameters for a range of values of Ra, and for a range of values of R. The results appear to support the following conclusions: (1) the value of a_1 generally decreases when Ra increases; and (2) the value of a_1 is smaller, and remains approximately constant when the separation R is deep inside the inertial range.

A good way to have a quick estimate of the importance of the nonlinear term [cf. Eq. (38)] is to measure $a_1\alpha\beta$, which is the coefficient of the nonlinear term. We see that this coefficient decreases significantly when we go from $\mathrm{Ra}=6.0\times10^8$ to $\mathrm{Ra}=5.8\times10^{14}$, becoming about 0.01 in the middle of the inertial range.

Next we show similar detailed calculations for the passive scalar data of Fig. 3. There is a slight complication since the pdf's $P(\Delta T,R)$ are not symmeteric in ΔT. Accordingly we need to fit separately the left and right branches of the distribution functions, and find the parameters α_+, α_-, β_+ and β_-, together with the appropriate values of $(a_1)_+$ and $(a_1)_-$. After doing all this we show the fits in Fig. 5. It appears to the eye that the quality of the fits is not significantly improved compared to the linear fit. To see this more clearly, in Table II we present the values of all the parameters involved in the fit. It is seen that the values of the parameters $a_{1_\pm}$ are close to zero, or, more precisely, $a_{1_\pm}=0\pm0.02$. The coefficient that measures the importance of the nonlinear correction, i.e., $a_1\alpha\beta$ is of the order of 10^{-4} for all the separations in the bulk of the inertial range.

TABLE II. Fitted parameters for the nonlinear fits in the passive advection data with Reynolds number 9.5×10^4. The probability density function $P(\Delta T,R)$ is asymmetric in this case, and we fit separately the left (minus subscript) and right (plus subscript) wings. All the parameters that measure the deviation from the linear fits are very small.

R	8	16	32	64	128	256	512	1024
β_-	0.94	1.12	1.28	1.54	1.77	1.85	1.76	1.86
β_+	1.15	1.32	1.56	1.63	1.75	1.82	1.87	1.82
$10^3(a_1)_-$	21	67	11	12	−3	9	10	−5
$10^3(a_1)_+$	18	14	−1	19	−6	6	0.6	12
$10^3(a_1\alpha\beta)_-$	10	16	3	1	−0.4	0.	0.2	−0.4
$10^3(a_1\alpha\beta)_+$	1.8	−0.7	−1.7	0.2	−0.3	−0.1	−0.2	0.4

IV. CONCLUSIONS

We presented a theoretical analysis of the relation between the two conditional averages $H(\Delta T,R)$ and $G(\Delta T,R)$ and the probability distribution function $P(\Delta T,R)$. The general relation is given by Eq. (17). From this relation it follows that if one of these averages factorizes to a function of ΔT times a function of R, the other cannot factorize as long as the distribution function does not factorize. The latter cannot factorize in any multiscaling statistics. Next we presented evidence that the conditional average $H(\Delta T,R)$ does factorize into a function of R times ΔT. This appears to be the case for both passive and active scalars when the Reynolds number is sufficiently large, and when R is in the bulk of the inertial range. The fusion rules which are believed to hold in a wide context furnish a prediction about the function of R that precedes ΔT in the conditional average, cf. Eq. (30). The data support the prediction of the fusion rules to very high accuracy. We do not at present have a theoretical *ab initio* derivation of the linear dependence that seems to be supported by the data. In view of the importance of this law for the study of the balance equation $D_n(R)=J_n(R)$ it seems to us that such a derivation is highly desirable.

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